Ample line bundles on a certain toric fibered 3-folds

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Abstract

Let X be a projective nonsingular toric 3-fold with a surjective torus equivariant morphism onto the projective line or a nonsingular toric surface not isomorphic to the projective plane. Then we prove that an ample line bundle on X is always normally generated.

Introduction

Let X be a projective algebraic variety and let L an ample line bundle on it. If the multiplication map

$$\mathrm{Sym}^k\Gamma(X, L) \longrightarrow \Gamma(X, L^{\otimes k}) \tag{1}$$

of the k-th symmetric power of the global sections of L to the global sections of the k-th tensor product is surjective for all positive integers k, then Mumford [4] calls L *normally generated*. A normally generated ample line bundle is always very ample, but not conversely.

If X is a toric variety of dimension n and L is an ample line bundle on it, then Ewald and Wessels [2] showed that $L^{\otimes k}$ is very ample for $k \geq n-1$, and Nakagawa [5] showed that the multiplication map

$$\Gamma(X, L^{\otimes k}) \otimes \Gamma(X, L) \longrightarrow \Gamma(X, L^{\otimes (k+1)})$$

is surjective for $k \geq n-1$. We know that there exists a polarized toric variety (X, L) of dimension $n \geq 3$ such that $L^{\otimes(n-2)}$ is not normally generated. We

also know that any ample line bundle on a nonsingular toric variety is always very ample (see [6, Corollary 2.15]). Ogata [7] showed that an ample line bundle L on a nonsingular toric 3-fold X is normally generated if the adjoint bundle $L + K_X$ is not big.

If a toric variety X of dimension $n \geq 2$ has a surjective torus equivariant morphism $\varphi : X \to Y$ onto a toric variety Y of dimension r $(1 \leq r < n)$ with connected fibers, then we call X a *toric fibered n-fold over Y*.

In this paper we restrict X to a certain class of toric fibered 3-folds.

Theorem 1 *Let X be a nonsingular projective toric fibered 3-fold over the projective line. Then an ample line bundle on X is always normally generated.*

Since a nonsingular toric surface not isomorphic to $\mathbb{P}^2$ has a torus equivariant surjective morphism onto $\mathbb{P}^1$, Theorem 1 imples the following corollary.

Corollary 1 *Let X be a nonsingular projective toric fibered 3-fold over a nonsingular toric surface not isomorphic to the projective plane. Then an ample line bundle on X is always normally generated.*

On a toric variety X of dimension n, the space $\Gamma(X, L)$ of global sections of an ample line bundle L is parametrized by lattice points in a lattice polytope P of dimension n (see, for instance, Oda's book[6, Section 2.2] or Fulton's book[3, Section 3.5]). If X has a surjective morphism $\varphi : X \to \mathbb{P}^1$ onto the projective line, then the corresponding polytope P has a special shape. From this fact, we shall prove Theorem 1.

In Section 3, we prove the same statement of Theorem 1 under the assumption that one invariant fiber of φ is irreducible. This is given as Proposition 3. Full satatement is proved in Section 4 as Proposition 4. In the end of this paper, we remark that nonsingularity condition is necessary by giving an example.

1 Toric varieties and lattice polytopes

In this section we recall the fact about toric varieties and ample line bundles on them and corresponding lattice polytopes from Oda's book [6] or Fulton's book [3].

Let $N \cong \mathbb{Z}^n$ be a free abelian group of rank n and $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual with the pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. By scalar extension to real numbers $\mathbb{R}$, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We also have the paring of $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ by scalar extension, which is denoted by the same symbol $\langle \cdot, \cdot \rangle$.

The group ring $\mathbb{C}[M]$ defines an algebraic torus $T_N := \mathrm{Spec}\mathbb{C}[M] \cong (\mathbb{C}^\times)^n$ of dimension n. Then the character group $\mathrm{Hom}_{\mathrm{gr}}(T_N, \mathbb{C}^\times)$ of the algebraic torus T_N coincides with M. For $m \in M$ we denote the corresponding character by $\mathbf{e}(m) : T_N \to \mathbb{C}^\times$.

Let Δ be a finite complete fan of N and $X(\Delta)$ denote the toric variety defined by Δ. Set $N_0 := \mathbb{Z}$ and $\Delta_0 := \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$. Then we have $X(\Delta_0) = \mathbb{P}^1$. If a surjective morphism $\varphi : X(\Delta) \to \mathbb{P}^1$ is torus equivariant, then it defines a morphism of fans $\varphi^\sharp : (N, \Delta) \to (N_0, \Delta_0)$. Moreover, if fibers of φ are connected, then $\varphi^\sharp(N) = N_0$. Set $N_0^\vee$ the dual to N_0. Then the dual homomorphism $\varphi^* : N_0^\vee \to M = N^\vee$ maps $N_0^\vee$ as a saturated submodule in M. Thus we have a direct sum decomposition $M \cong M' \oplus N_0^\vee$.

Set $N_f := (\varphi^\sharp)^{-1}(0)$ and $\Delta_f := \{\sigma \in \Delta; \varphi^\sharp(\sigma) = 0\}$. Then Δ_f is a fan of N_f and the toric variety $X(\Delta_f)$ is a general fiber of $\varphi : X(\Delta) \to \mathbb{P}^1$.

We define a *lattice polytope* as the convex hull $P := \mathrm{Conv}\{m_1, \ldots, m_r\}$ of a finite subset $\{m_1, \ldots, m_r\}$ of M in $M_{\mathbb{R}}$. We define the dimension of a lattice polytope P as that of the smallest affine subspace $\mathbb{R}(P)$ containing P.

Let X be a projective toric variety of dimension n and L an ample line bundle on X. Then there exists a lattice polytope P of dimension n such that the space of global sections of L is described by

$$\Gamma(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m), \tag{2}$$

where $\mathbf{e}(m)$ is considered as a rational function on X since T_N is identified with the dense open subset (see [6, Section 2.2] or [3, Section 3.5]). Conversely, a lattice polytope P in $M_{\mathbb{R}}$ of dimension n defines a polarized toric variety (X, L) satisfying the equality (2) (see [6, Chaper 2] or [3, Section 1.5]).

The k-th tensor product $L^{\otimes k}$ of L corresponds to the k-th multiple kP of P for a positive integer k. The condition that the multiplication map

$$\Gamma(X, L^{\otimes k}) \otimes \Gamma(X, L) \longrightarrow \Gamma(X, L^{\otimes (k+1)})$$

is surjective is equivalent to the equality

$$(kP) \cap M + (P \cap M) = ((k+1)P) \cap M.$$

A lattice polytope P in $M_{\mathbb{R}}$ is called *normal* if the equality

$$\overbrace{(P \cap M) + \cdots + (P \cap M)}^{k \text{ times}} = (kP) \cap M \tag{3}$$

holds for all positive integers k. This is equivalent to the condition that the equality

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M \tag{4}$$

holds for all positive integers k. We note that an ample line bundle L on a toric variety is normally generated if and only if the corresponding lattice polytope P is normal. We also note that the equality (3) holds if and only if for each lattice point $v \in (kP) \cap M$, there exists just k lattice points $u_1, \ldots, u_k$ in $P \cap M$ with $v = u_1 + \cdots + u_k$.

In order to prove the normality of lattice polytopes, the following theorem is useful.

Theorem 2 (Nakagawa [5]) *Let P be a lattice polytope in $M_{\mathbb{R}}$ of dimension n. Then we have the equality*

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M$$

for integer $k \geq n-1$.

From Theorem 2 we see that for the normality of P of dimension three, it is enough to show the equality

$$(P \cap M) + (P \cap M) = (2P) \cap M.$$

For a face of a lattice polytope, it is called an *edge* if it is of dimension one and a *facet* if of codimension one. A lattice polytope P of dimension n is called *simple* if at each vertex v, just n edges meet, that is, the convex cone $C_v(P) := \mathbb{R}_{\geq 0}(P - v)$ is written as

$$\mathbb{R}_{\geq 0} m_1 + \cdots + \mathbb{R}_{\geq 0} m_n$$

with $m_1, \ldots, m_n \in M$. Moreover, if the set $\{m_1, \ldots, m_n\}$ is a $\mathbb{Z}$-basis of M, then P is called *nonsingular*. For a face F of P, we call F is *nonsingular* if it is nonsingular with respect to the sublattice $\mathbb{R}(F) \cap M$. We note that a face of a nonsingular lattice polytope is also nonsingular.

2 Polygonal prisms

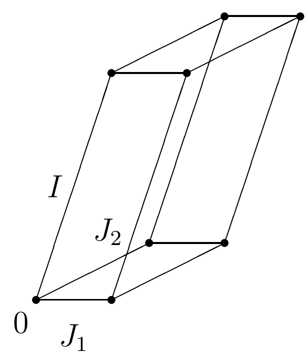
For two lattice polytopes P and Q in $M_{\mathbb{R}}$, we define the Minkowski sum as

$$P + Q := \{x + y \in M_{\mathbb{R}}; x \in P \text{ and } y \in Q\}.$$

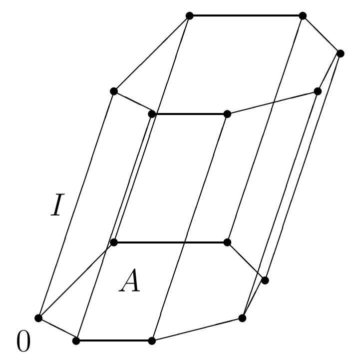
Then $P + Q$ is also a lattice polytope.

In this section we investigate the normality of a lattice polytope P of dimension three which is the Minkowski sum of a lattice polygon A of dimension two and a lattice line segment I. See Figure 1 (b). Here we set $M = M' \oplus L$, rank $M' = 2$, rank $L = 1$ and $A \subset M'_{\mathbb{R}}$.

If A is a parallelogram, that is, if A is the Minkowski sum $J_1 + J_2$ of two not parallel lattice line segments J_1 and J_2, then $P = A + I$ is normal because it is a parallelotope. See Figure 1 (a).



(a) $P = J_1 + J_2 + I$



(b) $P = A + I$

Figure 1: A polygonal prism P

From this observation we obtain the following proposition.

Proposition 1 *Let A be a nonsingular lattice polygon in $M'_{\mathbb{R}}$ not isomorphic to a basic triangle and I a lattice line segment not contained in $M'_{\mathbb{R}}$. Then $P = A + I$ is normal.*

In the case that A is a basic triangle, if $I \subset L_{\mathbb{R}}$, then $P = A + I$ is normal. Here a lattice triangle is called basic if it is isomorphic to the convex hull of the origin and a basis of $M' \cong \mathbb{Z}^2$.

In order to prove Proposition 1, it is enough to show the following lemma.

Lemma 1 *If a nonsingular lattice polygon $A \subset M'_{\mathbb{R}}$ is not basic, then it is covered by a union of lattice parallelograms.*

Proof. Take a coordinates (x, y) in $M'_{\mathbb{R}}$. Assume that a lattice polygon A is r-gonal. By a suitable affine transformation of M', we may take a vertex v_0 of A to be the origin, an edge E_1 from v_0 to be on the positive part of the x-axis and the other edge E_r from v_0 on the positive part of the y-axis. We will prove the lemma by dividing into several steps.

(a): The case of $A = \mathrm{Conv}\{0, (a, 0), (0, 1), (b, 1)\}$, that is, $r = 4$. If $a = b$, then A is a parallelogram. Set $a < b$ and $s = b - a$. If we set $A_i = \mathrm{Conv}\{0, (a, 0), (i, 1), (a + i, 1)\}$ for $i = 0, 1, \ldots, s$, then A_i is a parallelogram and A is covered by the union of A_i with $i = 0, \ldots, s$. The same is when $a > b$.

(b): Set $F(E_1) := A \cap (0 \leq y \leq 1)$. Since A is nonsingular, $F(E_1)$ is also a lattice polygon. From (a), we see that $F(E_1)$ is covered by a union of lattice parallelograms. For all edges $E_1, \ldots, E_r$ of A, define $F(E_i)$ in the same way. Then we have

$$A = A^o \cup \bigcup_{i=1}^{r} F(E_i),$$

where A^o is the convex hull of $\mathrm{Int}(A) \cap M'$.

If $\dim A^o \leq 1$, then A is covered by the union of $F(E_i)$. If $\dim A^o = 2$, then A^o is a nonsingular lattice polygon. If A^o is not isomorphic to a basic triangle, then we continue this process.

(c): When A^o is isomorphic to a basic triangle, we may consider A is the 4 times multiple $\mathrm{Conv}\{0, (4, 0), (0, 4)\}$ of a basic triangle, or, a polygon obtained from this by cutting off several basic triangles at vertices. Set $A' := A \cap (1 \leq y \leq 4)$. Then we have a decomposition $A = A' \cup F(E_1)$ and we see that A' is nonsingular and $\dim(A')^o \leq 1$. Thus we see that A is covered by a union of lattice parallelograms in this case.

Since the normalized area of A^o is an integer less than that of A, this process is stop after several steps. $\square$

Next we introduce another direct sum decomposition of $M = M' \oplus L$ with respect to I of the Minkowski sum $P = A + I$.

Set $L' := (\mathbb{R}I) \cap M \cong \mathbb{Z}$ and $M = M'' \oplus L'$ with the projection map $\pi : M \to M''$. We note that $M' = (\mathbb{R}A) \cap M \cong \mathbb{Z}^2$ does not always coincide with M''. Set $B := \pi(A) \subset M''_{\mathbb{R}}$. Then B is a lattice polygon in $M''_{\mathbb{R}}$.

Take coordinates (x, y) in $M''_{\mathbb{R}}$ and z in $L'_{\mathbb{R}}$. From a suitable affine transform of M, we may set so that a vertex v_0 of $P = A+I$ is the origin and P is contained in the upper half space $(z \geq 0)$. Then define $Q(A)$ as the convex hull of $B \times 0$ and $A + I$. The polytope $Q(A)$ is an upright polygonal prism with the r-gonal polygon B as its base and A as its roof. See Figure 2.

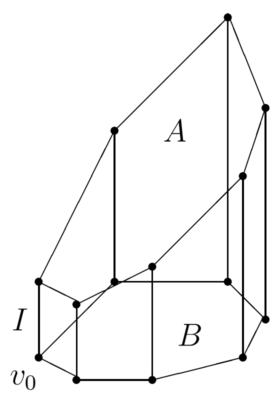


Figure 2: An upright polygonal prism $Q(A)$

Proposition 2 *Let A be a nonsingular lattice polygon in $M'_{\mathbb{R}}$ not isomorphic to a basic triangle. Then $Q(A)$ defined above is normal.*

Proof. Decompose the lattice polygon B into a union of basic lattice triangles B_i $(i = 1, \ldots, s)$ with vertices in $B \cap M''$. For each B_i, define $R(B_i)$ as the convex hull of $(B_i \times \mathbb{R}_{\geq 0}) \cap Q(A) \cap M$. Then the prism $R(B_i)$ is normal. We have a cover

$$Q(A) = (A + I) \cup \bigcup_{i=1}^{s} R(B_i).$$

Since $A + I$ is normal from Proposition 1, the polytope $Q(A)$ is normal. $\square$

3 Union of polygonal prisms

In this section we assume that a projective toric fibered 3-fold X over $\mathbb{P}^1$ has one irreducible invariant fiber.

As in Section 1, we set $N_0 := \mathbb{Z}$ and $\Delta_0 := \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$. Then $X(\Delta_0) = \mathbb{P}^1$. The torus equivariant morphism $\varphi : X = X(\Delta) \to \mathbb{P}^1$ is defined by the morphism of fans $\varphi^\sharp : (N, \Delta) \to (N_0, \Delta_0)$ with $\varphi^\sharp(N) = N_0$. Set $N_0^\vee$ the dual to N_0. Denote by L the image of the dual homomorphism $\varphi* : N_0^\vee \to M$. Then we have a direct sum decomposition $M = M_f \oplus L$, where $M_f^\vee \cong N_f := (\varphi^\sharp)^{-1}(0)$. The subset $\Delta_f := \{\sigma \in \Delta; \varphi^\sharp(\sigma) = 0\}$ is a fan of N_f. A general fiber of φ is the toric surface $X(\Delta_f)$.

Let $\mathcal{L}$ be an ample line bundle on a toric fibered 3-fold $X(\Delta)$ over $\mathbb{P}^1$. Let P be the lattice polytope in $M_\mathbb{R}$ coresponding to the polarized toric 3-fold $(X(\Delta), \mathcal{L})$. Denote by $\mathcal{L}_f$ the restriction of the ample line bundle $\mathcal{L}$ to $X(\Delta_f)$. The polarized toric surface $(X(\Delta_f), \mathcal{L}_f)$ defines a nonsingular lattice polygon $B \subset (M_f)_\mathbb{R}$. Then the lattice polytope P is contained in the polygonal prism $B \times \mathbb{R} \subset (M_f \oplus L)_\mathbb{R} = M_\mathbb{R}$ and each side wall of the prism contains a facet of P.

If one invariant fiber of φ is irreducible, then P has a facet isomorphic to B. We may draw the picture of P so that it is a polygonal upright prism with B as the base and the roof consists of a collection of lattice polygons. See Figure 3.

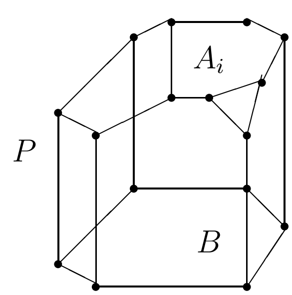


Figure 3: Union of upright polygonal prisms

Proposition 3 *Assume that a nonsingular projective toric fibered 3-fold $\varphi : X(\Delta) \to \mathbb{P}^1$ has one irreducible invariant fiber $\varphi^{-1}([1:0])$, that is, one*

irreducible invariant fiber is isomorphic to a general fiber. Then an ample line bundle on $X(\Delta)$ is always normally generated.

Proof. Take coordinates (x, y, z) in @ $M_{\mathbb{R}} = (M_f \oplus L)_{\mathbb{R}}$ so that $(M_f)_{\mathbb{R}} = (z = 0)$. Let $P \subset M_{\mathbb{R}}$ be a lattice polytope corresponding to an ample line bundle $\mathcal{L}$ on $X(\Delta)$.

From our assumption, P has the special facet B corresponding to the irreducible fiber of $\varphi^{-1}([1 : 0])$. From a suitable affine transform of M, we may assume that P is contained in the upper half space $(z \geq 0)$ and B is contained in the plane $(z = 0)$.

Set $A_1, \ldots, A_s$ the all facets in the roof of the upright prism P. Set $B_i = \pi(A_i)$ the lattice polygon in $(M_f)_{\mathbb{R}}$ defined as the image of a facet A_i by the projection $\pi : (M_f \oplus L)_{\mathbb{R}} \to (M_f)_{\mathbb{R}}$. For each facet A_i define $Q(A_i) := (B_i \times L_{\mathbb{R}}) \cap P$. Then we have a decomposition of P as a union of polygonal prisms $Q(A_i)$. For each A_i, set $M_i := (\mathbb{R}A_i) \cap M \cong \mathbb{Z}^2$. Then A_i is a nonsingular lattice polygon in $(M_i)_{\mathbb{R}}$.

If A_i is not a basic triangle, then $Q(A_i)$ is normal from Proposition 2. Even if A_i is a basic triangle if it meets a side wall of P, then it is normal because $M_i \oplus L \cong M$.

We assume that A_i is a basic triangle and meets no side walls of P. Set v_1, v_2, v_3 the three vertices of A_i and E_1, E_2, E_3 the edges of P from v_1, v_2, v_3 outside A_i, respectively. Let w_j be the lattice points on the edge E_j nearest v_j for $j = 1, 2, 3$. Set $\tilde{A}_i := \text{Comv}\{w_1, w_2, w_3\}$. Then the lattice triangle $\tilde{A}_i$ is similar and parallel to A_i since P is nonsingular. If $\tilde{A}_i \cong A_i$, then $P = Q(A_i)$. It contradicts the assumption. Thus $\tilde{A}_i$ is not basic. The subset $(\pi(\tilde{A}_i) \times L_{\mathbb{R}}) \cap P$ of P can be decomposed into a union of the slice $\text{Conv}\{A_i, \tilde{A}_i\}$ of the roof and the rest $Q(\tilde{A}_i)$. Both are normal.

Since P is covered by a union of normal lattice polytopes, it is normal. □

4 General case

Proposition 4 *Let $X(\Delta)$ be a projective nonsingular toric fibered 3-fold over $\mathbb{P}^1$. Then an ample line bundle on $X(\Delta)$ is always normally generated.*

Proof. As in the proof of Proposition 3, take coordinates (x, y, z) in @ $M_{\mathbb{R}} = (M_f \oplus L)_{\mathbb{R}}$ so that $(M_f)_{\mathbb{R}} = (z = 0)$. Let $P \subset M_{\mathbb{R}}$ be a lattice polytope

corresponding to an ample line bundle $\mathcal{L}$ on $X(\Delta)$. From a suitable affine transform of M, we may assume that P is contained in the upper half space $(z \geq 0)$.

Set $A_1, \ldots, A_s$ the all facets in the roof of the upright prism P and $B_i = \pi(A_i)$ for $i = 1, \ldots, s$. For a lattice point $m \in M$, denote by $l(m)$ the line through m parallel to the z-axis.

Take a lattice point m in $2P$. If m is located on the boundary of $2P$, then it is a lattice point on a lattice polytope of dimension less than three, hence, there exist two lattice points $m_1, m_2 \in (\partial P) \cap M$ such that $m = m_1 + m_2$.

We may assume that m is contained in the interior of $2P$. Then we may assume that the length of the line segment $l(m) \cap (2P)$ is bigger than two. In fact, if the length is two, then P is contained in the region $(k \leq z \leq k+1)$ and it is normal from Proposition 3 since all edges parallel to the z-axis, which are contained in side walls, have lengths bigger than or equal to two.

Moreover, we may assume that m is nearer the plane $(z = 0)$ than the center of the line segment $l(m) \cap (2P)$. If not, we may put P upside down. Set (a, b, c) the coordinates of m. Then we note $(a, b, c+1) \in 2P$.

Set $I := [0, (0, 0, 1)]$ the unit interval on the z-axis. By taking sufficiently large integer k, define the lattice polytope $\tilde{P} := (P + kI) \cap (0 \leq z \leq d)$ satisfies the condition of Proposition 3 for some integer d with $d \geq c/2$, that is, the facet $(P + kI) \cap (z = d)$ of $\tilde{P}$ coincides with B defined by a general fiber $(X(\Delta_f), \mathcal{L}_f)$.

Since m is a lattice point of $2\tilde{P}$, there exist $m_1, m_2 \in \tilde{P} \cap M$ with $m = m_1 + m_2$.

Let H be the plane containing two lines $l(m_1)$ and $l(m_2)$. Set (w, z) coordinates of H with respect to a basis of $H \cap M \cong \mathbb{Z}^2$ so that the sublattice $l(m_1) \cap M$ is the direct summand.

By a suitable choice of the w-coordinate, we may set $m_1 = (0, a)$, $m_2 = (d, b)$ and $0 < d$.

If d is even $2e$, then we may set $m_1 = (-e, a), m_2 = (e, b)$ and $m = (0, c)$ with $a + b = c$ by changing the w-coordinate again. Since $m \in 2(l(m) \cap P)$, we can fined $m'_1, m'_2 \in l(m) \cap P \cap M$ with $m = m'_1 + m'_2$.

If $d = 2e+1$, then we may set $m_1 = (-e, a), m_2 = (e+1, b)$ and $m = (1, c)$ with $a + b = c$. We can exchange m_1, m_2 with $m''_1 = (0, a), m''_2 = (1, b) \in \tilde{P} \cap M$ such that $m = m''_1 + m''_2$. Thus, we may set $d = 1$.

First, we assume that both two lines $l(m_1)$ and $l(m_2)$ meet one facet A_i. By changing a, b under the condition $a + b = c$, we may set $(0, a) \in P$ and $(0, a+1) \notin P$. We note $(1/2)m = (1/2, c/2) \in P$. For new a, b, set

$m'_1 = (0,a), m'_2 = (1,b)$. We note $l(m'_i) = l(m_i)$. Since $(1,c+1) \in 2P$, we know $(1/2,(c+1)/2) \in P$. Since two lines $l(m_1), l(m_2)$ are bounded by one facet A_i in P, they are bounded by a line in H. Since $(0,a) \in P$, $(1/2,(c+1)/2) \in P$ and $a+b=c$, we have $(1,b) \in P$. This implies $m'_1, m'_2 \in P \cap M$ with $m = m'_1 + m'_2$.

Next, we assume that two lines $l(m_1)$ and $l(m_2)$ do not meet the same A_i. Set A_j a facet meeting the line $l(m_1) + (1/2)m$ through the rational pont $(1/2)m$. We note $\pi((1/2)m) \in (1/2)M_f$. Set $\pi(m) = u \in M_f$ and $B_j := \pi(A_j)$. Decompose B_j into a union of basic triangles with vertices in the lattice M_f. Then the rational point $(1/2)u$ is contained in a basic triangle R. If $(1/2)u$ is a vertex of R, then there exists $m' \in P \cap M$ with $(1/2)m \in l(m')$. In this case, we can find $m'_1, m'_2 \in l(m')$ with $m = m'_1 + m'_2$. We assume $(1/2)u$ does not coincide with vertices of R. Since $(1/2)u \in (1/2)M_f$, the rational point $(1/2)u$ is the center of an edge of R. From this, we can choose lattice points $m_3, m_4 \in \tilde{P} \cap M$ with $m = m_3 + m_4$ such that two lines $l(m_3), l(m_4)$ meet one facet A_j. As in the same way of the previous paragraph, we can choose lattice points $m'_i \in P$ in the lines $l(m_i)$ so that $m = m'_3 + m'_4$. $\square$

Remark@ Nonsingularity condition of a toric fibered 3-fold $\varphi : X(\Delta) \to \mathbb{P}^1$ in Proposition 4 is necessary. We know a singular toric fibered 3-fold over $\mathbb{P}^1$ with a very ample but not normally generated line bundle on it. Finally, we will give an example found by Burns and Gubeladze [1, Exercise 2.24].

For a positive integer q, define a lattice tetrahedron as

$$Q_q := \mathrm{Conv}\{0, (1,0,0), (0,1,0), (1,1,q)\}.$$

If $q \geq 2$, then Q_q is not very ample. Set $I = [0,(0,0,1)]$ the unit interval on the z-axsis. Define $P_q := Q_q + I$ sa the minkowski sum. Let $(X, \mathcal{L})$ be the polarized toric 3-fold corresponding to P_q. Then this X is a singular toric fibered 3-fold over $\mathbb{P}^1$ and $\mathcal{L}$ is very ample. If $q \geq 4$, then $\mathcal{L}$ is not normally generated. See also [8].

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